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Abstract : The notion of Lie groups is introduced in many pure and applied mathematic questions. Originally created in the XIXth century by the norwegian mathematician Marius Sophus Lie, the theory was developed throughout the XXth century in parallel with the progress of algebra, topology and differential geometry. The Lie theory is a very active branch of mathematics with a strong impact on other areas.

Our objective in this thesis is to study the problem of deformation of discontinuous groups acting on homogeneous spaces. It is a matter of describing explicitly the parameter and deformation spaces of the actions of such groups. This description is essentially motivated by the study of some topological and geometrical properties of these spaces such as the stability and the Hausdorff properties that have been the subject of several works carried out during the last years in which a large developed part is about exponential solvable Lie groups. A space X is said to be homogeneous if there exists a Lie group G acting continuously and transitively on X. Let G be a Lie group, H a closed connected subgroup of G and Γ a discrete subgroup of G acting properly and freely on the homogeneous space G/H. Then the group Γ is discontinuous for G/H and the quotient space $\Gamma\backslash G/H$ is a Clifford-Klein form which is endowed with a smooth manifold structure. The problem of deformation consists in seeking how to deform Γ by means of homomorphisms from Γ to G (thus to consider the set $\mathrm{Hom}(\Gamma, G)$ of all these homomorphisms) in a way such that the deformed discrete group acts properly on G/H. The problem of describing deformations for Clifford-Klein forms was first advocated by Kobayashi in [22] for the general non-Riemannian setting and precisely involves as proposed in [25] the following issues :

(*i*) Describe the deformations of Γ inside G.

(*ii*) Accurately determine the set of deformation parameters in (*i*) that allow Γ to deform in a way to guarantee the proper discontinuity on G/H.

Towards these purposes, it is the parameters space introduced in [22]

$$\mathscr{R}(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{c} \varphi \text{ is injective and } \varphi(\Gamma) \\ \text{is discontinuous for } G/H \end{array} \right\} \quad (0.0.1)$$

rather than $\mathrm{Hom}(\Gamma, G)$ which plays a crucial role in this problem. In order to be precise on parameters, our main goal is to investigate the deforma-

tion space $\mathscr{T}(\Gamma, G, H)$ which is the quotient space of the parameter space given above by the equivalence relation of inner automorphisms. Deformation theory plays a crucial role in all branches of mathematics and physics. It is considered to be one fundamental guiding principle for developing the physical theory. The determination of the deformation spaces obtained by discontinuous groups acting on homogeneous spaces is a fundamental tool for the study of some geometrical structures of closed surfaces, such as projective structures, hyperbolic structures (Teichmüller spaces), complex structures etc. This problem was originally posed and studied by A. Weil in the case of Riemannian spaces [33] and by T. Kobayashi [22] in the case of non-Riemannian spaces. Under certain circumstances, the deformation spaces have been explicitly determinated and some of their differential and topological characteristics have been studied.
The major difficulty in the description of the parameter space is the characterization of the proper action of Γ on G/H. This question was considered by several authors. Let Γ be a closed subgroup of G. If G, H and Γ are reductive, T. Kobayashi showed in [21] that Γ acts properly on G/H if and only if the triple (Γ, G, H) satisfies the property of the compact intersection, denoted (CI). Lipsman conjectured that when G is connected simply connected nilpotent then the action of Γ on G/H is proper if and only if (Γ, G, H) satisfies the (CI) property. In [31] S. Nasrin proved this equivalence when G is two-step nilpotent. But in [35] T. Yoshino gave a counter-example when G is four-step nilpotent. In [12] A. Baklouti and F. Khlif extended the validity of this equivalence to the case when G is three-step nilpotent, G is special nilpotent, G is connected simply connected solvable and H or Γ is normal and G is exponential solvable and H or Γ is maximal.
When G is a completely solvable connected and simply connected Lie group, it is well known that any closed subgroup of G admits a syndetic hull, this result leads to a topological identification (algebraic characterization) of the parameter and deformation spaces (see [10]).

$$\mathscr{R}(\Gamma, G, H) \simeq \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim(\varphi(\mathfrak{l})) = \dim \mathfrak{l} \\ \exp(\varphi(\mathfrak{l})) \text{ acts properly on } G/H \end{array} \right\}$$

where $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ are respectively the Lie algebras of G, H and L, the syndetic hull of Γ and

$$\mathscr{T}(\Gamma, G, H) \simeq \left\{ \varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim(\varphi(\mathfrak{l})) = \dim \mathfrak{l} \\ \exp(\varphi(\mathfrak{l})) \text{ acts properly on } G/H \end{array} \right\} \Big/ G$$

where G acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by $g \cdot \varphi =_g \circ \varphi$.
The problem of stability is one of the most important problems in the deformation theory. It was introduced in [27] by Kobayashi-Nasrin and it may be

one fundamental genesis to understand the local structure of the deformation space. Let G be a Lie group, H a connected closed subgroup of G and Γ a discontinuous group for the homogeneous space G/H.

Définition 0.0.1. A homomorphism $\varphi \in \mathscr{R}(\Gamma, G, H)$ is said to be stable if there is an open set in $\text{Hom}(\Gamma, G)$ which contains φ and is contained in $\mathscr{R}(\Gamma, G, H)$.

As an application of the general theory, T. Kobayashi and S. Nasrin studied in [27] the setup of a properly discontinuous action of a discrete subgroup $\Gamma \simeq^k$ on $\mathbb{R}^{k+1} \simeq G/H$ through a certain two-step nilpotent affine transformation group G of dimension $2k+1$ when the connected subgroup H in question is $\mathbb{R}^k$. They explicitly determine the parameter and deformation spaces and characterize the set of non-stable morphisms. In [6] the authors studied the situation when G stands for the Heisenberg group and showed that the Hausdorff property of the deformation space is equivalent to the fact that $\mathscr{R}(\Gamma, G, H)$ is open in $\text{Hom}(\Gamma, G)$ (which means that the stability property holds). In [13] A. Baklouti and F. Khlif studied the case of the connected and simply connected threadlike Lie groups and in [19] F. Khlif treated the case of the connected threadlike Lie groups. Beyond the nilpotent case, L. Abdelmoula, A. Baklouti and I. Kedim studied in [1] the situation where G is solvable exponential and H is maximal in G and A. Baklouti, I. Kedim and T. Yoshino [11] studied the situation where G is an exponential Lie group and H contains $[G, G]$ or Γ is uniform in $[G, G]$. In [7] A. Baklouti, N. ElAloui and I. Kedim considered the setting where the underlying group G is two-step nilpotent, they proved first that if a pair (G, H) has the Lipsman property with G a connected simply connected nilpotent Lie group, then the parameter space is semi-algebraic. Also the authors provide an explicit description of the parameter and deformation spaces and established a stability and a Hausdorff theorems in the situation where G is two-step.
My work in this context is focused on the connected and simply connected nilpotent Lie groups. We first study the problem of deformations when the underlying group is three-step nilpotent. We provide a layering of Kobayashi's deformation space $\mathscr{T}(\Gamma, G, H)$ into Hausdorff spaces, which depends upon the dimensions of G-adjoint orbits of the corresponding parameter space. and we give a sufficient condition on (Γ, G, H) such that the topology of the deformation space is Hausdorff. Second, we prove a stability theorem for certain particular pairs (Γ, H). Further, A. Baklouti [4] proposed a new criterion of stability namely, the strong stability on layers making use of an explicit layering of $\text{Hom}(\Gamma, G)$ and we show that the stability and the Hausdorff properties hold with respect to this layering in the case when G is the Heisenberg group. Finally we generalize all results made on chapter 2 in the case when

G is an n-step nilpotent Lie group, $n \geq 3$. Let us now in a little more detail, describe the content of this thesis, which consists of four chapters.

Chapter 1 : "Preliminaries"

In this chapter, we recall the definitions and the concepts which we need afterwards and we fix some notations, terminologies and record some basic facts about deformations.

Chapter 2 : "Some problems of deformations on three-step nilpotent Lie groups"

In this chapter, we take the setting of three-step nilpotent Lie groups. We provide a stratification of both parameter and deformation spaces based on dimensions of the G-adjoint action of $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, where $\mathfrak{l}$ stands for the Lie algebra of syndetic hull of Γ. The main results are the following

Théorème 0.0.2. Let G be a three-step nilpotent connected and simply connected Lie group, H a connected subgroup of G and Γ a discontinuous group for the homogeneous space G/H. Then the parameter space $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is the disjoint union of two semi-algebraic sets $\mathscr{R}_1$ and $\mathscr{R}_2$ with $\mathscr{R}_1$ is a Zariski open in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$.

Théorème 0.0.3. There exist a G-invariant covering of $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ such that

$$\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})/G = \bigcup_{t=0}^{q} \bigcup_{t'=0}^{q'} \mathrm{Hom}^{t,t'}(\mathfrak{l}, \mathfrak{g})/G$$

as a union of Hausdorff subspaces.

Théorème 0.0.4. Let G be a three-step nilpotent connected and simply connected Lie group, H a connected subgroup of G and Γ a discontinuous group for the homogeneous space G/H. Then the deformation space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is written

$$\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \bigcup_{i=1}^{2} \bigcup_{t=0}^{q} \bigcup_{t'=0}^{q'} (\mathscr{R}_i \cap \mathrm{Hom}^{t,t'}(\mathfrak{l}, \mathfrak{g}))/G$$

as a union of Hausdorff subspaces.

Théorème 0.0.5. Let $G = \exp(\mathfrak{g})$ be a 3-step nilpotent Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G, Γ a discontinuous group for the homogeneous space G/H and $L = \exp(\mathfrak{l})$ its syndetic hull. If the dimensions of G-orbits in $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ and those in $p(\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}))$ are constant, then the deformation space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is a Hausdorff space.

Chapter 3 : "Stability of discontinuous groups acting on homogeneous spaces"

We study in this chapter some topological properties of the parameter space $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$, such as the stability. We establish a stability theorem on three-step nilpotent Lie groups. Then we prove that the stability may fail for general nilpotent Lie group of step ≤ 3. For this purpose, A. Baklouti [4] introduced the notions of stability on layers and strong stability on layers as follows

Définition 0.0.6. Let $\mathcal{L} = (H_i)_{i \in I}$ be a G-invariant covering of $\mathrm{Hom}(\Gamma, G)$. A homomorphism $\varphi \in \mathscr{R}_i := H_i \cap \mathscr{R}(\Gamma, G, H)$ is said to be stable on layers with respect to the layering $\mathcal{L}$, if there is an open set in H_i (which is not necessarily open in $\mathrm{Hom}(\Gamma, G)$) which contains φ and is contained in $\mathscr{R}_i$.

Définition 0.0.7. The parameter space $\mathscr{R}(\Gamma, G, H)$ is said to be strongly stable on layers with respect to the layering $\mathcal{L} = (H_i)_{i \in I}$, if each of its elements is stable on layers with respect to $\mathcal{L}$ and if $(\mathscr{T}_i(\Gamma, G, H) = \mathscr{R}_i(\Gamma, G, H)/G)_{i \in I}$ is a covering of $\mathscr{T}(\Gamma, G, H)$ as a union of Hausdorff subspaces.

The main result is the following

Théorème 0.0.8. Let G be the Heisenberg group, H a connected subgroup of G and Γ a discontinuous group for the homogeneous space G/H. Then there exists a layering of $\mathrm{Hom}(\Gamma, G)$ for which the parameter space $\mathscr{R}(\Gamma, G, H)$ is strongly stable on layers.

Chapter 4 **: "Some problems of deformations on nilpotent Lie groups"**
Our objective in this chapter is to generalize all the previous results obtained in chapter 2 to the general framework of nilpotent Lie groups.

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